Transport Process Considerations

- Relative dimensions.
- Typical analyte concentrations.
- Determining sample size.
- Characteristics of fluids.
- Reynolds Number and Laminar flow.
- Hagen-Poiseuille flow or Poiseuille flow.
- Surface area to volume.
- Diffusion.

Length & Volume Comparisons

Determining Sample Volume

- The relationship between sample volume \( V \) and analyte concentration is shown here:

\[
V = \frac{1}{\eta A_i C_i}
\]

where:
- \( \eta \) (eta) is the sensor efficiency \( 0 \leq \eta \leq 1 \),
- \( N_A = 6.02 \times 10^{23} \), or Avogadro's number, and
- \( A_i \) is the concentration of analyte, \( i \).

- The higher the concentration of analyte and/or better the sensor, the less the volume of sample that is needed.

Example: \( K^+ \) (Potassium) in Serum

Potassium has a concentration of 3.5 to 5.3 mmol/liter in serum. Assume a sensor efficiency of 0.1, and concentration of 3.5 mmol/liter, approximately what is the smallest volume needed:

\[
\text{Volume} = \frac{1 \times 6.02 \times 10^{23}}{0.1 \times 3.5 \times 10^{-3} \text{mol/L}} \approx 47 \times 10^{-10} \text{ liter}
\]
Sample Requirements for Detection…

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What is a Fluid?

A fluid is a substance that deforms continuously under the application of shear (tangential) stress of any magnitude.

Newtonian fluid – shear force is directly proportional to the rate of strain.

- This includes most fluids and gasses.

Adapted from Nguyen, NT and ST Wereley, Fundamentals and Applications of Microfluidics, Artech House, Boston, MA (2002).

Some Definitions: Viscosity…

where

\[ \tau = \mu \frac{\partial u}{\partial y} \]

Kinematic Viscosity…

- Kinematic viscosity of a fluid relates the absolute viscosity to density:

\[ \nu = \frac{\mu}{\rho} \]

where

- \( \nu \) (nu) is the kinematic velocity (m/s),
- \( \mu \) (mu) is the absolute viscosity, and
- \( \rho \) (rho) is the density or mass per unit volume;

Specific Weight & Specific Gravity…

- The specific weight of a fluid is defined as the weight per unit volume:

\[ \gamma = \rho g \]

where

- \( \gamma \) (gamma) is specific weight,
- \( \rho \) (rho) is the density or mass per unit volume,
- \( g \) is the local acceleration due to gravity;

- The specific gravity (SG) of a fluid is the ratio of the density of the fluid to the density of water at some specified temperature:

\[ SG = \frac{\rho}{\rho_{\text{H}_{2}O\text{ at } 4^\circ C}} \]

Flow Considerations

- Continuum assumption
  - Fluid characteristics vary continuously throughout the fluid.
  - May not be true with certain molecular content.
- Laminar vs. transitional and turbulent flow.
  - Based on Reynolds number.
- Fluid kinematics.
  - Field representation
- Hagen-Poiseuille flow (pronounced pwähz'-wèuhl)
- Poiseuille’s law.
- Surface area to volume.
- Diffusion
Factors Influencing Flow...

- Kinematic properties - velocity, viscosity, acceleration, vorticity.
- Transport properties - viscosity, thermal conductivity, diffusivity.
- Thermodynamic properties - pressure, thermal conductivity, density.
- Other properties - surface tension, vapor pressure, surface accommodation coefficients.

Reynolds Number

- In circular tube flows without obstruction, conventional fluid mechanics would dictate that Reynolds numbers smaller than about 2100 typically indicate laminar flow, while values greater than 4000 are turbulent. In between is transitional.
- Ratio between inertial and viscous forces:

\[ Re = \frac{\rho V D}{\mu} \]

where
- \( \rho \) (rho) is the fluid density,
- \( V \) is the mean fluid velocity,
- \( D \) is the pipe diameter and
- \( \mu \) (mu) is the fluid viscosity.

Example: Reynolds Number

- In a channel carrying water (viscosity of 10^{-3} kg/(s m), density of 10^3 kg/m^3), in a channel with diameter of 10 \( \mu \)m, at a velocity of 1 mm/s, the Reynolds number is 10^{-2}:

\[ Re = \frac{10^3 \text{ kg/m} \times 10^{-3} \text{ m/s} \times 10^{-6} \text{ m}}{10^{-2} \text{ m} / \text{sec}} = .01 \]
Contrast this to a channel with diameter of 100 µm and fluid velocity of 10 m/s, where the Reynolds number is a 1000:

\[
Re = \frac{\frac{10^2 \text{kg}}{\text{m} \cdot \text{s}} \times \frac{0.1 \text{m}}{10^3 \text{m/s}}}{10^{-3} \frac{\text{kg}}{\text{m} \cdot \text{s}}} = 1000
\]

With channels < 1 mm in width and height, and velocities not greater than cm/s range, the Reynolds number for flow will be so low that all flow will be laminar.
- In laminar flow, velocity of a particle is not a random function of time.
- Useful for assays and sorting by particle size, and allows for creation of discrete packets of fluid that can be moved around in a controlled manner.

Microfluidic Flow Challenges
- Flows at low Reynolds numbers pose a challenge for mixing. Two or more streams flowing in contact with each other will mix only by diffusion.
- As surface area to volume increases surface tension forces become significant, leading to nonlinear free-boundary problems.
- As suspended particle sizes approach the size of the channel, there is a breakdown of the traditional constitutive equations.

Fluid Kinematics
- A field representation of flow is the representation of fluid parameters as functions of spatial coordinates and time.
- The velocity field is defined as:

\[
\mathbf{V} = \mathbf{u}(x,y,z,t)\hat{i} + \mathbf{v}(x,y,z,t)\hat{j} + \mathbf{w}(x,y,z,t)\hat{k},
\]

where \( u, v, w \) are the \( x, y, \) and \( z \) components of the velocity vector, and \( t \) is time.

- For problems related to flow in most microchannels, we can consider that one or two of the velocity components will be small relative to the others, and reduce the problem to 1D or 2D flow.
Position Vector…

- Particle location is in terms of its position vector \( \mathbf{r}_A \):

The velocity of a particle is the time rate of change of the position vector for that particle:

\[
\frac{d\mathbf{r}_A}{dt} = \mathbf{V}_A
\]

The direction of the fluid velocity relative to the \( x \) axis is given by:

\[
\tan \theta = \frac{v}{u}
\]


“Streamlines”…

- A streamline is a line that is everywhere tangent to the velocity field. In steady flows, the streamline is the same as the path line, the line traced out by a given particle as it flows from one point to another.

For a 2D flow the slope of the streamline must be equal to the tangent of the angle that the velocity vector makes with the \( x \) axis:

\[
\tan \theta = \frac{v}{u}
\]

\[
\frac{dy}{dx} = \frac{v}{u}
\]

If the velocity field is known as a function of x and y, this equation can be integrated to give the equation of the streamlines.

The notation for a streamline is:

\[ \psi \text{ (psi) = constant on a streamline} \]

The stream function is:

\[ \psi = \psi(x, y) \]

Various lines can be plotted in the x-y plane for different values of the constant. For steady flow, the resulting streamlines are lines parallel to the velocity field.

Uniform Flow...

The streamlines are all straight and parallel, and the magnitude of the velocity is constant.

Streamline Coordinates...

- Example of a steady 2D flow.
- \( 's' \) is a unit vector along the streamline.
- \( 'n' \) is normal to the streamline.
- Velocity is always tangent to the 's' direction.
Flow Around a Circular Cylinder...

“Flow Net” and the Velocity Potential φ...

Hagen-Poiseuille Flow (pwiːz- wɛl̬i-)

- Hagen-Poiseuille flow or Poiseuille flow is the steady, incompressible, laminar flow through a circular tube of constant cross section. The velocity distribution is expressed as:

\[ v_z = \frac{1}{4\mu} \left( \frac{\partial p}{\partial z} \right) \left( r^2 - R^2 \right) \]

where
- \( \mu \) (mu) is the fluid viscosity,
- \( \frac{\partial p}{\partial z} \) is the z component of the pressure gradient,
- \( r \) is the distance from the center of the tube, and
- \( R \) is the radius of the tube.
Velocity Distribution...

- The velocity distribution is parabolic at any cross sectional view:

  ![Parabolic Velocity Distribution](image)

  - Each part of the fluid can be visualized as moving along its own path line.
  - The maximum velocity is at the pipe center.
  - The minimum velocity (zero) is at the pipe wall.
  - Shear stress is the consequence of the velocity variation and fluid viscosity.

Poiseuille’s Law...

- Poiseuille’s law describes the relationship between the volume rate of flow, $Q$, passing through the tube and the pressure gradient:

$$Q = \frac{\pi D^4 \Delta p}{128\mu L}$$

where:

- $Q$ is the volume rate of flow,
- $D$ is the diameter of the tube,
- $\Delta p$ is pressure drop over length $L$ along the tube,
- $\mu$ is the fluid viscosity, and
- $L$ is the radius of the tube.

- For a given pressure drop per unit length, the volume flow rate is inversely proportional to the viscosity and proportional to the tube radius to the fourth power.
- Doubling of the tube radius produces a sixteen-fold increase in flow.

The mean velocity is:

$$V = \frac{Q}{\pi R^2} = \frac{R^2 \Delta p}{8\mu L}$$

and the maximum velocity occurring at the center of the tube is:

$$v_{max} = 2V.$$
Surface Area to Volume

- Surface area to volume (SAV) increases significantly as dimensions are reduced for microfluidic channels. For example, for a circular microchannel 100 µm in diameter, the SAV ratio is:

\[
\text{SAV} = \frac{2\pi r}{\pi r^2 L} = \frac{2}{r} \times 4 \times 10^9 \text{ m}^{-1},
\]

where
\[
\begin{align*}
& r \text{ is the radius, and} \\
& L \text{ is the length.}
\end{align*}
\]

Diffusion

- In laminar flow, two or more streams flowing in contact with each other mix only by diffusion.

\[
x^2 = 2Dt,
\]

where
\[
\begin{align*}
& x \text{ is the distance a particle moves,} \\
& t \text{ is the amount of time, and} \\
& D \text{ is the diffusion coefficient.}
\end{align*}
\]

\[
D = \frac{RT}{\sqrt{\pi}N_A} \left[ 1 + C \left( \frac{RT}{\eta} \right)^{1/2} \right],
\]

where
\[
\begin{align*}
& R \text{ is the gas constant,} \\
& T \text{ is temperature,} \\
& \eta \text{ is particle radius,} \\
& N_A \text{ is Avogadro’s number,} \\
& C \text{ is concentration in moles/liter,} \\
& \eta \text{(syr)} \text{ is the solution viscosity, and} \\
& \gamma \text{ is the activity coefficient in moles/liter.}
\end{align*}
\]

Example: Diffusion of Hemoglobin

- How long does it take hemoglobin to diffuse 1 cm in water (D=10⁻⁹ cm² s⁻¹)?

\[
t = \frac{(1 \text{ cm})^2}{2 \times 10^{-9}} = 5 \times 10^4 \text{ (56 days!)}
\]

- How long to diffuse 10 µm (.001 cm)?

\[
t = \frac{(10^{-5} \text{ cm})^2}{2 \times 10^{-9}} = 5
\]
The optimal size domain for microfluidic channel cross sections is somewhere between 10 µm and 100 µm. At smaller dimensions detection is too difficult and at greater dimensions unaided mixing is too slow. Therefore, the cross sectional area for a square channel with width of 50 µm will be approximately 2.5 x 10^-3 mm². The flow range will be 1 to 20 nL/sec. When diluting an assay component, the two flows must be controlled within ~1%, or pL/sec range.

Summary

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