

Mathematical Analysis of Frog Jumping and
Calculation of Power Development

by

Steven S. Saliterman

Introduction

Photographic techniques used to study a frog's motion when jumping are now supplemented by electronic measurement of the acceleration and flight times, and of the jump range. Initial velocity and angle of trajectory at the start of flight are determined with the assistance of a nonlinear equation root-finder computer program (1).

Described here is the development of two real, simultaneous, nonlinear equations that are constructed from data from the Laser Activated Amphibian Monitor System (2), and from a center of mass-determining photographic exercise.

Power development in the leg extensor muscles of the jumping frog is calculated from the initial velocity, height the frog reaches when flight begins, mass of the frog and his leg extensor muscles, and from the acceleration time.

Previous Works on Frog Jumping

Although power development in the jumping frog has been previously unexplored, many attempts have been made to relate the distance an amphibian jumps to physical dimensions of the

animal. Rand and Rand (3,4) have determined that the total body length of *Bufo Marinus* varies with the cube root of the weight and with the hind limb length. Moreover, they have shown that the distance jumped varies directly with the hind limb length. Stokely and Berberian (5) were unable to conclude that "changes in the relative proportions of the skeletal parts of the frog hind leg appreciably influence jumping ability." Gans and Rosenberg (6) determined a linear relationship between body weight of *Bufo Marinus* and the estimated force exerted when the animal jumps. (Gans and Parsons (7) have given a rather interesting analysis on the origins of the jumping mechanism in frogs.)

Gray's work on animal locomotion, Hirsch's work on the mechanism of frog jumping, and E.C.B. Hall-Cragg's analysis of the jump of the Less-Galago are perhaps the most useful works in regard to power development in the muscles of the jumping frog. Gray (8,9) in his discussion of amphibian jumping correctly describes the movement of the center of mass of the frog from the time it is in flight to the instant it touches the ground. His mathematical interpretation of the event is however limited. Nevertheless, his photographs do clearly show the orientation of the limbs during flight. Hirsch (10) has done an excellent analysis of the acceleration phase. Figure 1 shows the form of the frogs acceleration. Figure 2 shows from a side and top view the orientation of the limbs during the various stages of Figure 1. The drawings in Figure 1 were made by Hirsch via high speed

photography. Former techniques used to measure the initial velocity of a jumping frog consisted of measuring the path over which the frog accelerated and the time spent accelerating. The actual path taken by the center of mass can only be estimated. In addition, the acceleration is not uniform and leads to complications when trying to determine the initial velocity. Nevertheless, photographs do give an accurate measure of the angle of trajectory. Developing photographs for each jump of an experiment consisting of many frogs and many jumps would be expensive and time consuming.

Hirsch's work paired with Zugs (11) work on the relation between jumping and osteometrics may eventually allow for calculation of the change in length individual muscles undergo during the acceleration phase. Given work development of a particular jump the technique used by E.C.B. Hall-Craggs (12) may be used to determine the force and tension developed in individual muscles. This technique consists of determining the force each muscle must apply to given joints so that the resultant of all forces produces the final measured force. Given Hirsch's side and top views of the limb orientation as acceleration proceeds, the angular velocity with which the various joints "open up" could be calculated. This would allow for calculation of the velocity of shortening of individual muscles. If the changes in velocity, tension, and length of individual muscles in the accelerating frog were determined, then it is possible to construct a three-dimensional representa-

tion of the chemical energy available for contraction (13).

The Frog Jump Model

The various phases of a frog's jump are shown in Figure 3. Acceleration begins as the frog begins to exert a force against the ground. Judging from Hirsch's photographs of the acceleration phase, the center of mass rotates forward as it progresses upward, and comes into line with the force vector between the frog's feet and earth (the dashed line is the pathway of the c.o.m.). However, the pathway is uncertain and it is unwise to assume a uniform acceleration or to estimate distance traveled by the c.o.m. during this phase. Once the frog begins flight (c.o.m. is at point b) the pathway taken is parabolic until deceleration begins. Air friction is negligible and does not noticeably vary the parabolic flight (14). As deceleration continues the c.o.m. travels forward as the outstretched arms absorb the impact, and then falls back to point e as the frog's legs come to rest on the ground.

The range of a jump, or segment R in Figure 3, and the time of flight, t_f can be measured electronically. The initial velocity, V_0 and angle of trajectory, θ_0 are determined by solving for the root of the following derived equations:

$$\begin{aligned} \text{I. } 0 &= t_f V_0 \cos \theta_0 + H \cos \theta_0 + M \cos (|\theta_w|) - D - R \\ \text{II. } 0 &= H \sin \theta_0 - t_f V_0 \sin \theta_0 + 0.5gt_f^2 - M \sin (|\theta_w|) \end{aligned}$$

Where $\theta_w = \arctan ((V_0 \sin \theta_0 - gt_f)/(V_0 \cos \theta_0))$

The values for H, M and D are measured by a c.o.m. - determining photographic exercise. This procedure is done only

once for each test frog and is detailed in Figure 4. The gravitational acceleration constant g , is 980 cm/sec^2 . Θ_w is the reentry angle. Note in Figure 3 that the distance traveled to the summit (point b to c) by the center of mass in flight is less than the distance traveled coming down from the summit (point c to d). Therefore, the reentry angle will be steeper and the reentry velocity greater than the initial angle and velocity.

Equation (I) above was derived from the following:

a. Segments $A + E + N = D + R$ (in Figure 3)

$$\text{or } 0 = A + E + N - D - R$$

b. $A = H \cos \Theta_0$

$$E = t_f V_0 \cos \Theta_0$$

$$N = M \cos (|\Theta_w|)$$

$D =$ Premeasured distance from the tips of the toes to tips of the fingers measured in the ZY - plane.

$R =$ Electronically measured range of the jump. Distance from tips of fingers before jump to tips of fingers after jump. (In reality, LAAMS measures a different yet equal distance, the distance from pt. f to pt. g.)

c. Therefore: $0 = H \cos \Theta_0 + t_f V_0 \cos \Theta_0 + M \cos (|\Theta_w|) - D - R$

Equation (II) above was derived from the following:

a. Segments $L + J = I$

$$\text{or } 0 = I - L - J$$

$$\begin{aligned}
 \text{b.} \quad L &= M \sin (|\Theta_w|) \\
 J &= t_f V_o \sin \Theta_o - 0.5 g t_f^2 \\
 I &= H \sin \Theta_o
 \end{aligned}$$

$$\begin{aligned}
 \text{c. Therefore: } 0 &= H \sin \Theta_o - t_f V_o \sin \Theta_o + 0.5 g t_f^2 - \\
 &M \sin (|\Theta_w|)
 \end{aligned}$$

The reentry angle is simply the arctangent of the ratio of the Y-velocity component to the X-velocity component at t_f .

$$\text{III. } \Theta_w = \arctan ((V_o \sin \Theta_o - g t_f) / (V_o \cos \Theta_o))$$

Equations (I) and (II) can be solved by the Nonlinear Equation System Root-finder computer program by Kenneth M. Brown (15), by a program written by this author (16), or by any equivalent program available in many computer center libraries.

The max. height reached by the frog is $I + S$, where S equals:

$$\text{IV.} \quad V_o^2 \sin (2 \Theta_o) / (2g)$$

The reentry velocity is the square root of the sum of the squares of the Y and X velocity components at t_f :

$$\text{V.} \quad V_w = ((V_o \sin \Theta_o - g t_f)^2 + (V_o \cos \Theta_o)^2)^{0.5}$$

Introduction to Power Development

Power development (p.d.) is defined as the total energy expended by the leg extensor muscles as they contract, divided by the product of the time over which this energy was expended (the acceleration time) and the mass of the extensor muscle groups.

$$\text{VI. power development} = \text{total energy expended} / (\text{acceleration time} \times \text{mass of extensor muscles})$$

The acceleration time begins with the first sign of force against the ground and ends when the distal phalanges leave the ground.

The total energy expended includes energy seen as useful work, and energy that is wasted in the form of heat. The mechanical efficiency of muscle is the ratio of the work performed by a muscle to the total energy spent (17).

$$\text{VII. efficiency} = \frac{\text{energy expended on work}}{\text{total energy expended}}$$

Measurement of energy expended in the form of heat is difficult to perform in the isolated muscle preparation, and nearly impossible to measure directly in the jumping frog. In fact, the total amount of heat liberated when muscle contracts has been the center of considerable controversy. Hill (18,19, 20,21) contends that when muscle contracts energy is liberated in three forms: 1) A, or heat of activation, 2) ax, or heat of shortening, and 3) W, the mechanical work.

$$\text{VIII. total energy} = W + A + ax$$

The amount of shortening is "x," and "a" is a constant. In a tetonic contraction the heats of activation due to successive stimuli are together known as the maintenance heat.

Carlson, Hardy and Wilkie (22) propose that shortening heat as such does not exist. Their technique for measuring total energy output is based on measurement of phosphocreatine hydrolysis. Phosphocreatine hydrolysis did show a constant term that corresponds to "A" above and a linear term that

corresponds with "w."

Rearrangement of equations (VI) and (VII) above reveals that the total energy expended (equation VI) can be replaced by:

IX. energy expended on work/efficiency

The efficiency of the leg muscle system is difficult to determine. Individual muscle fibers will have an efficiency of about 20% (23). For a similar group of frogs (species, age, sex) differences in p.d. will be more a function of muscle mass and neuromuscular intactness, rather than structural differences giving different muscle-bone lever efficiencies. Strictly speaking, absolute p.d. can not be determined unless the efficiency of the muscle system is known. However, relative differences in p.d. for a frog forced to jump several times consecutively, and differences found pre and post exposure to an experimental variable are vitally significant. If efficiency is considered similar for similar groups of frogs, then relative differences in p.d. may be calculated. An approximate figure for p.d. in a single jump may be obtained by using 20% for efficiency.

Calculation of Power Development

Calculation of p.d. requires the following data:

1. t_a , acceleration time.
2. V_o , initial velocity.
3. M_f , mass of the frog.
4. M_m , mass of the contracting extensor muscles.

5. h , vertical displacement of the center of mass from rest to the end of acceleration.

$$h = H \sin \theta_0 - Q = I - Q$$

(See Figure 4 for measurement of Q)

6. g , gravitational acceleration constant = 980 cm/sec^2 .

The total energy expended for useful work equals:

$$X. (M_f g h) + (0.5 M_f V_0^2)$$

Power development equals:

$$XI. ((M_f g h) + (0.5 M_f V_0^2)) / (0.2 \text{ to } M_m)$$

Summary

Calculation of power development in the leg extensor muscles of the jumping frog requires knowledge of the initial velocity of the jump, height which the frog reaches when flight begins, mass of the frog and his leg extensor muscles, and the acceleration time. Time of flight and range are measured rapidly and economically with the Laser Activated Amphibian Monitor System. Two nonlinear simultaneous equations (I, II) are developed for calculation of initial velocity from the LAAMS data, and from a c.o.m. determining photographic exercise carried out once for each test frog. Acceleration time is also measured electronically. Mass of the frog is determined before each jump on a triple beam balance. After an experiment the frog's leg extensor muscles are dissected and weighed. Total energy expended for useful work (X) and power development (XI) are then calculated. Because of assumptions made on efficiency, relative differences in p.d. are more significant than absolute measurements.

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Fig. 1. The spring-like thrust of acceleration is easily seen in Hirsch's high-speed photographs of the jump.

Fig. 2. These side and aerial views of the bones and joints during the acceleration phase correspond with the individual frames of figure 1. The corresponding bones from the pelvic girdle downward are the femur, tibio-fibula, astragalus/calcaneum, metatarsals, and phalanges. A complete discussion of the bones and muscles of the hind limb can be found in Dr. Alexander Ecker's The Anatomy of the Frog, Oxford, 1889.

Fig. 3. The Frog Jump Model. Points a,b,c,d and e are locations of the center of mass at rest, start of flight, summit, start of deceleration, and at rest again respectively. Pts. h and j are the tips of the toes, and pts. i and k are the tips of the fingers (as measured in the Z-Y plane). The distance between pts. f and g is equal to the distance between i and k, and is represented by the segment R. D is the distance between the tips of the toes and the tips of the fingers. Segments H and M are measured by a c.o.m.-determining photographic exercise described in figure 4. The initial velocity V_0 , and angle of trajectory θ_0 , are found by solving two nonlinear simultaneous equations derived by manipulation of segments I,J,L,A,E,D,N and R (see text).

I is a vertical segment from pt. b to the distance axis. J is a vertical segment from pt. m to d. L is a vertical segment from pt. d to the distance axis. Segment groups H, I, A

and L, M, N each form right triangles. E is the horizontal displacement of the c.o.m. during flight, and S (pt. c to e) plus I is the maximum height obtained.

Fig. 4. Determination of center of mass in the resting and extended frog. An anesthetized frog is tied in a "rest" configuration with light thread. The frog is then suspended by the midline and photographed at two different suspension points. Overlapping the photographic negatives reveals where the center of mass lies in the ZY-plane. Segment Q is the distance from the c.o.m. to the apparent ground. Similar procedures are carried out with the frog's legs fully extended, and again, this time with the arms also fully extended. The second sequence gives H (figure 3) and the third sequence gives M (figure 3). The latter two sequences are accomplished by fastening a thin rod on the back of frog in the midline. Individual photos are taken by sliding the frog to a different position on the centrally suspended rod.

Note that I (figure 3) minus Q, is the vertical distance the c.o.m. is carried during the acceleration phase.

WALTER HIRSCH'S STUDY
OF THE ACCELERATION
PHASE

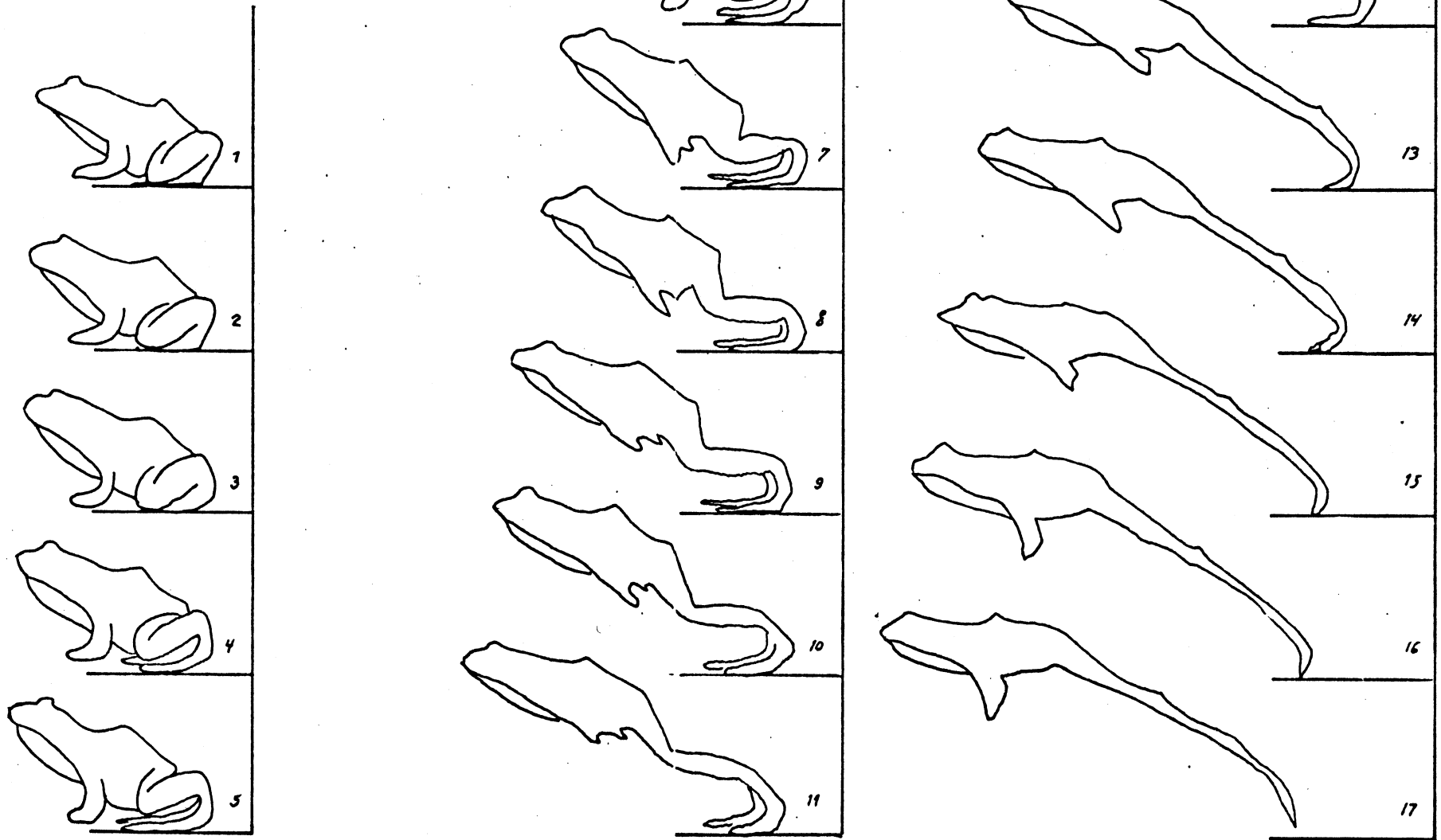


Fig. 1

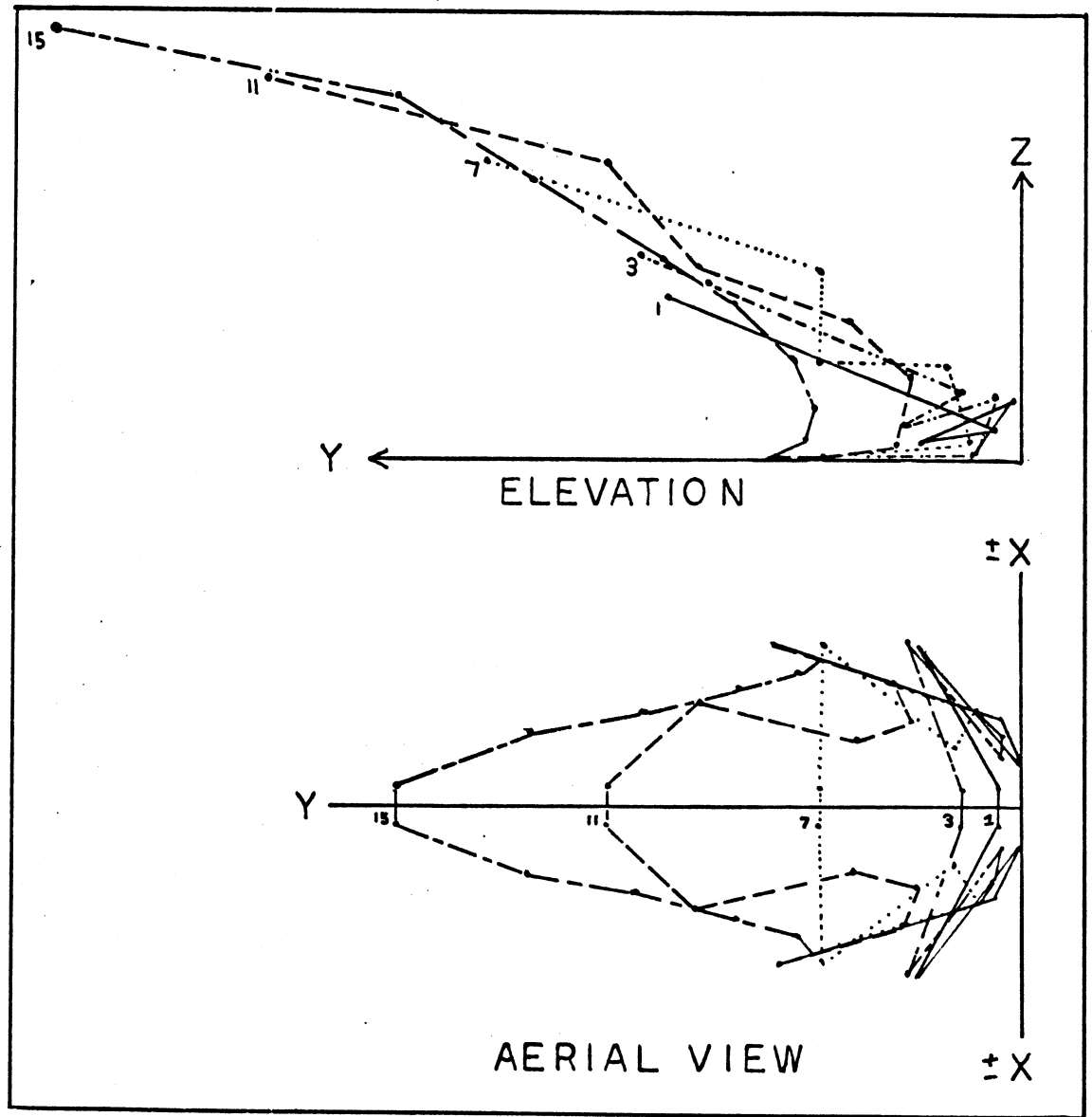
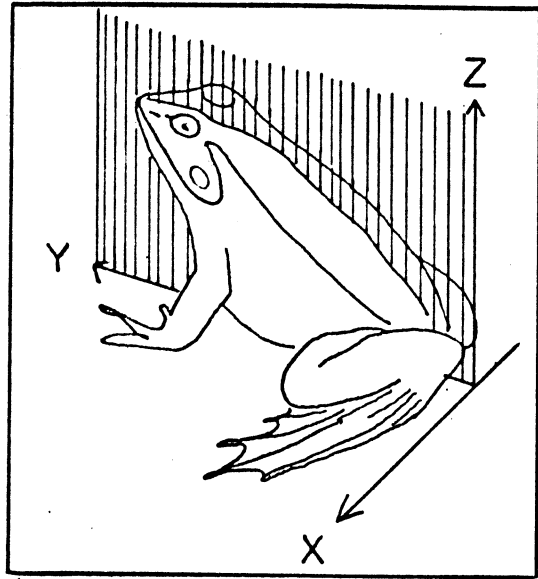


Fig. 2.

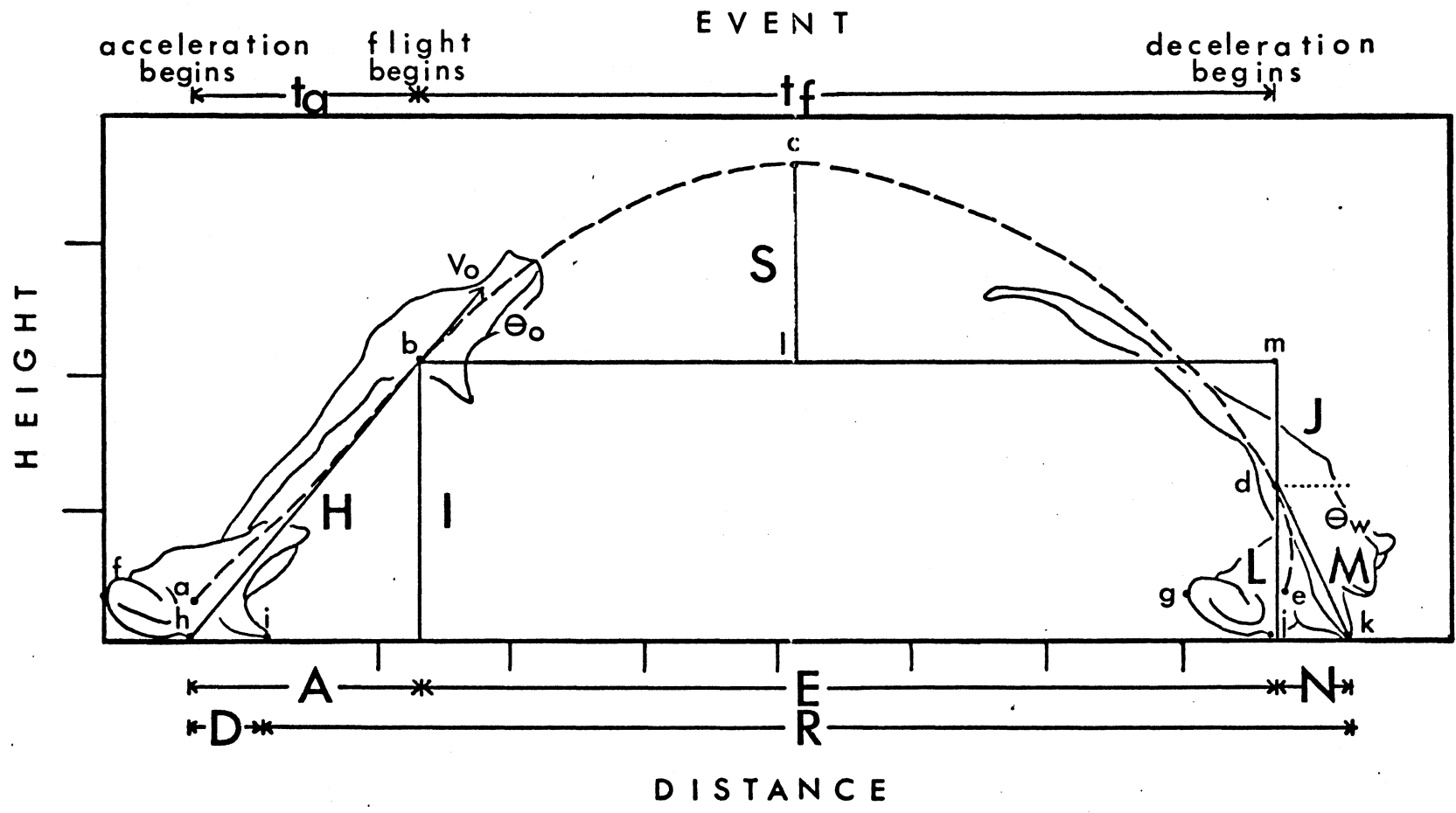


Fig. 3.

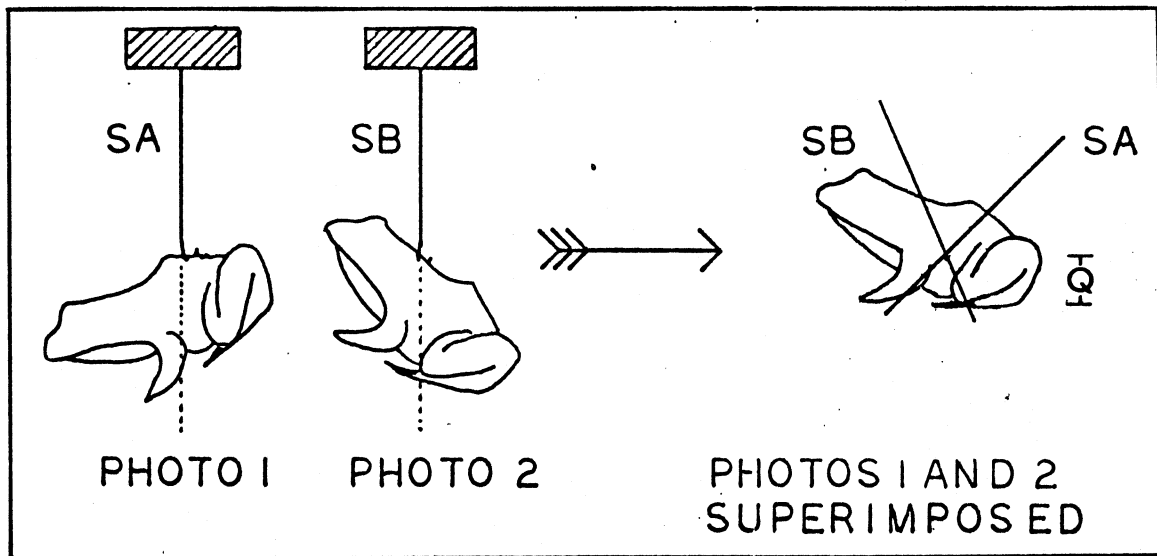


Fig. 4.